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Theory and Methodology

ZAPROS-LM – A method and system for ordering multiattribute alternatives

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Abstract: A method to aid in qualitative evaluation of multiattribute alternatives is proposed. It not only elicits information from a decision-maker in a qualitative form but tries to use it without resort to numbers, and to apply rational logic for comparison of alternatives. Special procedures for identification of possible inconsistencies in decision-maker's information and elimination of them in a dialogue with a decision-maker are developed. Possibilities for verification and explanation of the results for partial ordering of a large set of alternatives are shown. Two main assumptions are used: transitivity of the decision-maker's preferences and preferential independence of attributes. Problems of justification of these properties in real tasks of decision making are discussed. The description is accompanied by an example.

Keywords: Decision theory; Multiple criteria; Behaviour; Project management; Planning

1. Introduction

There exist many different methods and systems which are supposed to aid in decision making in the case of many criteria. The peculiarity of these methods is that all of them need to use information (or judgements) from decision makers revealing their preferences. Only this information makes it possible to find a compromise between conflicting objectives and yield a good decision.

One of the most popular approaches in this field is that of multiattribute utility theory (MAUT) which is often substituted by multiattribute value theory in practical tasks (Keeney and Raiffa, 1976; Watson and Buede, 1987). This approach requires quantitative estimation of weights for criteria and scores (values) for each alternative upon each criterion. This information is then used to assess a value for each alternative by combining weights and scores upon separate criteria on the basis of some scalar function (most often, an additive one). Over the years decision making was largely viewed as determining an appropriate aggregation rule (see Keeney and Raiffa, 1976; Rivett, 1977; Svenson, 1979, 1983).

Still, certain difficulties occur in the application of these methods, especially in the ill-structured problems (Montgomery, 1977). First, they are due to the necessity to obtain complicated judgements from decision makers, concerning weights and scores. There also exist facts about limitations in human

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beings' possibilities in evaluation and comparison of multiattribute options. These limitations lead to inconsistencies in people's judgements (Tversky, 1969; Russo and Rosen, 1975; Hoffman et al., 1968), or to implementation of simplified rules, which do not consider some essential aspects of options under consideration (Payne, 1976; Larichev and Moshkovich, 1988; Montgomery, 1977). For the second, they make it necessary to use quantitative measures even for qualitative concepts. But it is not always necessary. In this article a method for comparison of multiattribute alternatives is described. It is based on the same axioms as the multiattribute value function theory, but makes it possible sometimes to compare alternatives without resort to quantitative judgements or scaling of qualitative ones.

2. Problem formulation

There exists a rather large class of practical problems in which it is necessary to rank-order (at least partially) the alternatives. The constructed alternatives' order may be used, e.g. to fund as many of the best projects as we can in such tasks as portfolio selection (Clarckson, 1979; Furems and Moshkovich, 1984), or to reject the least preferable alternatives from a production plan (Zuev et al., 1980; Larichev, 1982), or to define a limited group of the most preferred alternatives (Kirkwood and Sarin, 1985; Furems et al., 1982) and so on.

In cases when we have many alternatives (may be tens or even hundreds) it may be considered logical enough to construct some rules on the basis of decision-maker (DM) preferences in the criteria space, and to use this set of rules for rank-ordering of the set of real alternatives. Just this problem is under further consideration.

Let us assume that alternatives are estimated upon a set of criteria. Criterion scales are discrete and have verbal formulations of quality grades (Larichev, 1979). It is also assumed that values upon the criterion scale are ordered from the most to the least preferable one for a DM (an example of such criteria for the task of job selection is given in Appendix A). So, the problem may be formulated as follows:

Given:

1. $K = \{q_i\}$, $i = 1, 2, \dots, Q$ – A set of criteria.
2. n_q = Number of possible values on the scale of the q-th criterion ($q \in K$).
3. $X_q = \{x_{iq}\}$ – A set of values for the q-th criterion rank-ordered from the most to the least preferable one (the ordinal scale of the q-th attribute); $|X_q| = n_q$ ($q \in K$).
4. $Y = X_1 \times X_2 \times \cdots \times X_Q$ – A set of vectors $y_i \in Y$ of the following type: $y_i = (y_{i1}, y_{i2}, \dots, y_{iQ})$, where $y_{iq} \in X_q$; $N = |Y| = \Pi_{q=1}^{Q} n_q$.
5. $A = \{a_i\} \subseteq Y$ – A set of vectors describing the real alternatives.

Required: to form an ordering of multiattribute alternatives of the set A on the basis of the decision-maker's preferences.

3. An approach to the problem solution

As it has been mentioned above, we will try to construct a rule for pairwise comparison of vectors from Y (on the basis of decision maker's preferences) and to apply this rule for comparison of vectors from A. Let us introduce a binary relation $R \subseteq Y \times Y$, reflecting relationship of preference or indifference in pairs of vectors from the set Y. Further the following properties of the relation R will be used:

Definition 1. R is called *reflexive* if $(y_i, y_i) \in R$, $\forall y_i \in Y$.

Definition 2. R is called *connected* if $\forall y_i, y_j \in Y$, $(y_i, y_j) \in R$ or $(y_j, y_i) \in R$.

Definition 3. R is called *symmetric* if $\forall y_i, y_j \in Y$, $(y_i, y_j) \in R$ implies $(y_j, y_i) \in R$.

Definition 4. R is called *antisymmetric* if $\forall y_i, y_j \in Y$, $(y_i, y_j) \in R$ and $(y_j, y_i) \in R$ implies $i = j$.

Definition 5. R is called *transitive* if $\forall y_i, y_j, y_k \in Y$, $(y_i, y_j) \in R$ and $(y_j, y_k) \in R$ implies $(y_i, y_k) \in R$.

Definition 6. R is called *quasi-order* if R is reflexive and transitive.

Definition 7. R is called a *linear quasi-order* if R is reflexive, transitive and connected.

Definition 8. P is called a *strict preference relation* if it is antisymmetric and transitive.

Definition 9. I is called an *indifference relation* if it is symmetric and transitive.

The most popular approach to solution of such a task is the construction of a scalar value function $v(y)$ with the following usual properties:

$$v(y_i) > v(y_j) \quad \text{if } (y_i, y_j) \in P,$$

$$v(y_i) = v(y_j) \quad \text{if } (y_i, y_j) \in I.$$

If a value function exists, then it induces a complete ordering on the set Y. Conversely, if we do have a complete preference order on the set Y, then we are able to construct a value function by merely attaching any increasing sequence of numbers to vectors arranged in the increasing order. The idea of many methods (see, e.g. Watson and Buede, 1987) is to find out (with the help of a value function), preferences which we find difficult to articulate directly, using only those preferences that we can express easily. But the opinion of what is easy for people, is not the same for different groups of scientists.

Larichev et al. (1987) tried to collect and classify all elementary operations in information processing used in normative decision making. Twenty-three such operations were defined and analyzed for their complexity for human beings. The primary conclusion of that study was that quantitative evaluation and comparison of different objects is much more difficult for subjects than conducting the same operations using qualitative (ordinal) expressions.

This conclusion is currently popular, and is based not only on the data of descriptive investigations, but also on experience in real decision problems. So, we propose to rank-order multiattribute alternatives according to the decision-maker's preferences, elicited in a simple way, ensuring reliability of the obtained information.

Different versions of the method for ordering multicriteria alternatives were published in Larichev et al. (1974, 1979), Gnedenko et al. (1986) under the name ZAPROS ('closed procedure near references situations'). In this paper we present a modified method which use some ideas of the previous versions.

The main idea of the approach described below is based on the concept of joint ordinal scale built according to the DM's preferences. The joint ordinal scale (JOS) means that all possible values upon all criteria are ranked-ordered for the DM upon his (or her) preferences. This ordinal scale may be effectively used for comparison of real alternatives.

As values upon each criterion scale are ordered for a DM, a relation of strict preference (or dominance) P^0 on the set Y may be defined:

$$P^0 = \{(y_i, y_j) \in Y \times Y \mid \forall q \in K,\ v_{iq} \text{ is not less preferable than } y_{jq}$$
$$\text{and } \exists q^0 \text{ such that } y_{iq^0} \text{ is more preferable than } y_{jq^0}\}.$$

Let us now analyze what relation may be constructed on the set Y if the joint ordinal scale is formed. Let R be a linear quasi-order built on the set $X = \{X_q\}$, $q = 1, 2, \ldots, Q$, according to the DM's preferences. It is clear that the relations of strict preference P^0, defined on values of separate criteria, are part of R (as they also reflect the DM's preferences). We are able to rank-order elements from X according to R (this ranking will be the joint ordinal scale). Note that some elements from X will occupy the same place in the ranking, as R is reflexive (that is there may be values equally preferable for the DM).

As now we know all the relations between values upon different criteria we are able to introduce the following binary relation of quasi-order on Y:

$$R^1 = \{(y_i, y_j) \in Y \times Y \mid \forall y_{iq_1}(q_i \in K) \exists y_{jt(q_1)}(t(q_1) \in K) \text{ such that } (y_{iq_1}, y_{jt(q_1)}) \in R \text{ and if } q_1 \neq q_2, \text{ then } t(q_1) \neq t(q_2)\}.$$

It is clear that $P^0 \subset R^1$.

Thus, the rule for comparison of vectors from Y (and from the set A accordingly) may be formulated as follows:

Vector $y_i \in Y$ is not less preferable than vector $y_j \in Y$ if for each component of the vector y_i there exists a component of the vector y_j with not more preferable value upon joint ordinal scale (binary relation R).

The introduced rule will not guarantee us the comparability of any two vectors from Y, but this is a result that corresponds with the accuracy of the input judgments (Roy, 1985).

So, the task is to construct such joint ordinal scale on the basis of the DM's preferences. Further, it will be shown that to do this, we will need simple ordinal pairwise comparisons, fulfilled by the DM for some vectors from Y, differing in values upon not more than two criteria. To prove the correctness of the introduced rule, two rather simple assumptions about the properties of a decision-maker's preference system may be used: preferential independence of criteria (Fishburn, 1970; Keeney and Raiffa, 1976) and transitivity of the resulting preference-indifference relation (Mirkin, 1974).

In the next two sections the way to elicit valid information on the DM's preferences is described. Verification of the transitivity of the DM's preferences is carried out, possible violations are corrected. In the following section foundations for the correctness of the implementation of JOS for comparison of vectors from Y are given. After that the possibilities for checking preferential independence of criteria for the DM are discussed. Recommendations for task modification are given in the case of dependency. Formal representation is illustrated on the examples for the task of job selection.

4. Elicitation of information on DM's preferences

A number of descriptive studies (see Larichev and Moshkovich, 1988; Russo and Rosen, 1975; Tversky, 1969; Larichev et al., 1988; Payne, 1976; Montgomery, 1977) show that people are more consistent in pairwise comparisons of multiattribute alternatives if the alternatives differ in values by not more than two or three criteria.

The construction of the joint ordinal scale will be carried out in the following way. We have ordinal scales (see Appendix A for example) in which values are ordered from the most preferable (first values) to the least preferable ones. To make the description easier we shall use sometimes numerical indicators of verbal values in formal representations and verbal values in a dialogue with a DM. So, for the example we have six criteria with values 1, 2 and 3 for each (these criteria may be used in the task of job selection).

To construct the joint ordinal scale we must compare pairs of different values upon different criteria. As the values upon other criteria may influence the comparison result, we propose to compare vectors from Y which have all the same values but two. The number of such pairs from Y may be very large. Therefore it was proposed to compare vectors near two reference situations, as it will be shown that this information is enough to construct the JOS.

Each vector from Y (that is a combination of values upon criteria) is an image of a certain alternative for a DM. The two most bright 'contrasting' images correspond to the combinations of the best and the worst values upon all criteria. Such vectors were called reference situations (Larichev et al., 1979).

Definition 10. Vectors with all the best or all the worst values upon all criteria will be called reference situations.

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        You are to compare the following alternatives:

 1. The salary is rather high.
 2. There are nice possibilities for training.
 3. There are good possibilities for promotion.

                     ALTERNATIVE 1

 4. Type of the job position is not appropriate.
 5. Location of the job is almost ideal.

                     ALTERNATIVE 2

 4. Type of the job position is almost ideal.
 5. Job is located far away from where you want.

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                     Possible answers:

 1. Alternative  1  is more preferable than alt. 2 .
 2. Alternative  1  and  2  are equally preferable.
 3. Alternative  2  is more preferable than alt. 1.

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Figure 1. Visualization of alternatives for comparison near the first reference situation

Definition 11. Let us say that L is a list of vectors near a reference situation if L is a subset of vectors from Y with all components except one equal to those of this reference situation.

Let us form lists L_1 and L_2 near the first and the second reference situation correspondingly:

$$L_1 = \{y_i \in Y \mid y_{iq} = x_{q1} \ \forall q \neq s(q \in K),\ y_{is} \neq x_{s1},\ s = 1, 2, \ldots, Q\},$$

$$L_2 = \{y_i \in Y \mid y_{iq} = x_{qn_q} \ \forall q \neq s(q \in K),\ y_{is} \neq x_{sn_s},\ s = 1, 2, \ldots, Q\}.$$

It is clear that $|L_1| = |L_2| = N_1 = \Sigma_{q=1}^{Q}(n_q - 1)$.

We propose to carry out an interview with a DM for each list of vectors. The procedure will be described for the list L_1 as an example. An interview near the second reference situation (list L_2) is carried out in the same way.

The DM is asked to compare pairs of vectors from the list L_1. All questions necessary to compare all vectors from the list L_1 are asked (an example of such a question is presented in Figure 1). The results of pairwise comparisons by a DM may be presented in a form of binary relations as follows:

1) P_{DM} is a set of pairs $(y_i, y_j) \in L_1 \times L_1$ if according to a DM's opinion y_i is more preferable than y_j, or if $(y_i, y_j) \in P^0$.
2) I_{DM} is a set of pairs $(y_i, y_j) \in L_1 \times L_1$ if according to a DM's opinion y_i and y_j are equally preferable, or when $i = j$.

As DM's answer, 'vector y_i is more preferable than vector y_j' means that $(y_i, y_j) \in P_{DM}$ and $(y_j, y_i) \notin P_{DM}$; the relation P_{DM} is antisymmetric.

As DM's answer, 'vector y_i and vector y_j are equally preferable' means that $(y_i, y_j) \in I_{DM}$ and $(y_j, y_i) \in I_{DM}$; the relation I_{DM} is reflexive and symmetric.

If we require the transitivity of relations P_{DM} and I_{DM}, then according to Mirkin (1974) the relation $R_1 = P_{DM} \cup I_{DM}$ is a linear quasi-order on the set L_1.

So, we need to construct a transitive binary R_1 relation on L_1. To do this we must be sure that the DM's information is consistent. But, as we know, in any interview with a DM there is a possibility of errors in his (her) responses. These errors may be random or may occur while comparing similar in quality alternatives. Therefore, in the information, elicited from a DM inconsistencies (contradictions) may appear. A special procedure for detection and elimination of contradictions in DM's answers is proposed (Moshkovich, 1988).

5. Elimination of intransitivity in DM's responses

In the problem under consideration the possible contradictions in DM's answers may be determined as violations of transitivity of relations P_{DM} and I_{DM} (and in general as violations of transitivity of R_1).

In general the problem of detection and elimination of intransitivity in pairwise comparisons is rather complicated. It is analogous to the task of cycles' elimination in a graph (Wilson, 1972; Kendall, 1969; Aho et al., 1962; Ore, 1962). It is known that the problem of determination of the minimal number of arcs necessary to be destroyed in a graph to make it acyclic is a NP-complete problem (Garey and Johnson, 1979). This means that in nowadays it is considered that this problem can not be solved exactly in a polynomial time. That is why there are works, devoted to the development of an approximate solution of this problem (Aho et al., 1974; Ore, 1962).

Our problem of detection and elimination of intransitivities in the information, received from a DM, has two peculiarities which make the above mentioned traditional approach ineffective.

First, the elimination of arcs in a graph may lead to a partial loss of information on vectors' comparisons which is undesirable. Secondly, our task is to detect the contradictory DM's responses which have led to cycles, but not to find the minimal number of arcs to be eliminated.

The main idea of the proposed approach (Moshkovich, 1988) is as follows. It is based on the assumption of the transitivity of DM's preferences and considers violations of this assumption to be errors in DM's responses.

The transitivity of preferences assumes that if:

1) $(y_i, y_j) \in P_{\mathrm{DM}}$, then $\forall y_k \in L_1$ and $(y_j, y_k) \in P_{\mathrm{DM}} \Rightarrow (y_i, y_k) \in P_{\mathrm{DM}}$.
2) $(y_i, y_j) \in I_{\mathrm{DM}}$, then $\forall y_k \in L_1$ and $(y_j, y_k) \in I_{\mathrm{DM}} \Rightarrow (y_i, y_k) \in I_{\mathrm{DM}}$.
3) $(y_i, y_j) \in P_{\mathrm{DM}}$, then $\forall y_k \in L_1$ and $(y_j, y_k) \in I_{\mathrm{DM}} \Rightarrow (y_i, y_k) \in P_{\mathrm{DM}}$.
4) $(y_i, y_j) \in I_{\mathrm{DM}}$, then $\forall y_k \in L_1$ and $(y_j, y_k) \in P_{\mathrm{DM}} \Rightarrow (y_i, y_k) \in P_{\mathrm{DM}}$.

Therefore after each comparison of vectors from L_1 made by a DM, this information may be extended on the basis of transitivity (transitive closure of the binary relation defined on the set L_1 is being built).

After that the DM is presented with the next pair of vectors from L_1, for which the relation has not been defined. When the DM's response is obtained, the transitive closure is developed and the procedure is maintained up to the moment of establishing relations for all pairs from L_1.

Statement 1. *If $R_{\mathrm{DM}} = P_{\mathrm{DM}} \cup I_{\mathrm{DM}}$ is transitive and $(y_i, y_j) \notin R_{\mathrm{DM}}$, then the transitive closure R^*_{DM} of the relation $R'_{\mathrm{DM}} = R_{\mathrm{DM}} \cup (y_i, y_j)$ will be transitive for any type of the DM's response on comparison of y_i and y_j.*

Proof. Evident, because a DM is presented only with pairs of vectors from $L_1 \times L_1$, for which previous responses have not predefined any relation. So, any variant of the response $((y_i, y_j) \in P_{\mathrm{DM}}; (y_i, y_j) \in I_{\mathrm{DM}}; (y_j, y_i) \in P_{\mathrm{DM}})$ will not contradict previous responses.

Once the response is received, transitive closure of the newly obtained relation is being built. It is known that transitive closure of the acyclic graph does not lead to cycles (Aho et al., 1974). So, we can say that such a procedure does not lead to intransitivity of the relation being built.

To test responses of a decision maker, we suggest to present the DM with additional pairs of vectors for comparison on the basis of the following principle: *The relation between each pair of vectors from L_1 is*

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                ALTERNATIVE  1

 Type of the job position is almost ideal.
 Location of the job is some distance from the ideal.
 The salary is rather high.
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                ALTERNATIVE  2

 Type of the job position is good enough (in field).
 Location of the job is almost ideal.
 The salary is rather high.
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                ALTERNATIVE  3

 Type of the job position is almost ideal.
 Location of the job is almost ideal.
 The salary is on the average level.
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  / alternatives have the best values upon other criteria/

Earlier you said that  alt.1 was equal to  alt.2, alt.2 was
more preferable than alt.3. That's why alt.1 is more preferable
than alt.3. Now you say that alt.3 is equal to alt.1.

     What comparison would you like to change ?


     Possible   answers  :   1 2,  2 3,   1 3.

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Figure 2. Visualization of DM's contradictory responses

to be defined directly (by a DM's response) or indirectly (by transitive closure) no less that two times. This requirement means that if a DM by two of his (her) responses (may be indirectly – by transitive closure) has equally defined the relation between vectors from L_1 in some pair, then this relation is considered to be proven. If the relation between vectors from L_1 in some pair has been defined only once and only upon transitive closure, then this pair is presented additionally to a DM for comparison.

If the DM's response does not conflict with the previously obtained information, then the judgment is considered to be correct. If there is some difference, the triple of vectors for which a pairwise comparison contradicts the transitivity of the relation being built on L_1, is found out: that is of vectors $y_i, y_j, y_k \in L_1$ such that one of the following statements is fulfilled:
1) $(y_i, y_j) \in P_{DM}$; $(y_j, y_k) \in I_{DM}$; $(y_i, y_k) \in I_{DM}$.
2) $(y_i, y_j) \in P_{DM}$; $(y_j, y_k) \in P_{DM}$; $(y_i, y_k) \in I_{DM}$.
3) $(y_i, y_j) \in P_{DM}$; $(y_j, y_k) \in P_{DM}$; $(y_k, y_i) \in P_{DM}$.
4) $(y_i, y_j) \in P_{DM}$; $(y_j, y_k) \in I_{DM}$; $(y_k, y_i) \in P_{DM}$.

Such a triple may always be detected, because after each of DM's responses we have built transitive closure of the obtained relation. In this case the DM is asked to reconsider the situation and to change one (or more) of his (or her) previous responses to eliminate intransitivity (example of such situation is presented in Figure 2).

After the corrected responses are obtained, they are incorporated into the information on the DM's preferences as follows.

It is supposed that we only start the interview with a DM (that is we have only these three responses for pairwise comparisons of y_i with y_j; y_j with y_k; y_i with y_k). At this time we also know that these responses do not contradict each other. We assign each of these responses to P_{DM} or I_{DM} accordingly and carry out the transitive closure of the obtained relation as in the initial interview with a DM. Subsequently we carry out further formation of the binary relation on L_1, but the information for pairwise comparisons is obtained from *the previous responses of a DM.*

RANK	JOINT ORDINAL SCALE (ordered values)	Vector
1	Type of the job position is almost ideal. Location of the job is is almost ideal. The salary is rather high. There are nice possibilities for training. There are good possibilities for promotion.	11111
2	There are normal possibilities for training.	11121
3	There are almost none possibilities for training.	11131
4	Type of the job position is good enough (in field Location of the job is some distance from ideal. There are moderate possibilities for promotion.	21111 12111 11112
5	The salary is on the average level.	11211
6	Job is located far away from where you want. The salary is a bit lower the average level.	13111 11311
7	There are almost no possibilities for promotion.	11113
8	Type of the job position is not appropriate.	31111
9	The salary is rather poor	11411

Figure 3. Ordering of values upon all criteria scales

This way guarantees that previous DM's responses do not contradict the newly built relation (as we use only responses for those pairs of vectors for which previous responses have not predefined some relation). As a result we obtain new transitive relation on the set L_1 in which the necessary changes have been made, but all previous responses not contradictory to the new ones are maintained unchanged. After that, the condition of 'double test' for each pairwise comparison is checked for this new information.

The proposed approach makes it possible to form an effective procedure for an interview with a DM to build the required relation, as the redundancy of the obtained information is limited to a reasonable condition of minimal necessary test for DM's responses. Some experiments with ZAPROS with students showed that for the introduced task of job selection (see Appendix A) the number of questions was in the range of 8 to 25, averaging at 16. Our experience in real tasks showed us that usually it takes not more than an hour to build the JOS for 7–8 criteria. In Appendix B the reader can find some estimations for the number of questions to the DM in the worst case.

6. Implementation of information on the DM's preferences

As a result of an interview with a DM and transforming his (her) responses to a non-contradictory variant the relation $R_1 = P_{\mathrm{DM}} \cup I_{\mathrm{DM}}$ of linear quasi-order on the set L_1 is built. Information on comparison of vectors' pairs obtained near the first reference situation may be used for construction of the joint ordinal scale (Zuev et al., 1974; Ozernoy and Gaft, 1978). The joint ordinal scale in these works was considered to be the ranking of the set of vectors near the first reference situation (see the example of such scale for the task of job selection in Figure 3).

If we recall that vectors near the first reference situation differ from that reference situation in only one component, we can consider the place, obtained by the vector in this ranking to be the place of this unique component in the JOS. In this sense the JOS ranks not criteria values by themselves, but rather the 'trade-offs' between values of different criteria. While comparing a pair of vectors from the list L_1, the DM responded in fact to the question: "What is more preferable for you – the decreasing in value from the best to the level i upon criterion 1 or the decreasing in value from the best to the level j upon criterion 2?".

This information leads to the rule for comparison of vectors from Y introduced in Section 3. The correctness of the rule may be proved for the case of pairwise preferential independence of all criteria (Gnedenko et al., 1986, Larichev and Moshkovich, 1991). Let us remind the idea of preferential independence.

Definition 12. Criteria s and t of the set K are preferentially independent from the other criteria of this set, if preference between vectors with equal values upon all criteria but s and t does not depend on the values of the equal components.

Now, maintaining the formal adequacy of the statement, given in Gnedenko et al. (1986), let us formulate this in a more precise way.

Let $L'_1 = L_1 \cup (x_{11}, x_{21}, \dots, x_{Q1})$ (L_1 complemented by a vector with all the best values). The relation R_1 is complemented by relations which reflect the preference of the vector $(x_{11}, x_{21}, \dots, x_{Q1})$ to all other vectors from L_1 and its equality to itself. Then the statement may be formulated in the following way.

Statement 2. *If each pair of criteria from K ($Q \geq 3$) does not depend preferentially on other criteria, then vector $y_i = (y_{i1}, y_{i2}, \dots, y_{iQ}) \in Y$ is not less preferable for the DM than vector $y_j = (y_{j1}, y_{j2}, \dots, y_{jQ}) \in Y$ if $\forall$ criterion $s \in K$, $\exists$ criterion $t(s) \in K$ such that*

$$(x_{11}, x_{21}, \dots, x_{(s-1)1}, y_{is}, x_{(s+1)1}, \dots, x_{Q1})\ R_1\ (x_{11}, x_{21}, \dots, x_{(t-1)1}, y_{jt}, x_{(t+1)1}, \dots, x_{Q1}),$$

and if $s \neq q$, then $t(s) \neq t(q)$.

Proof. Given in Appendix C.

This statement guarantees the correctness of the rule, introduced earlier for comparison of vectors from the set Y in the case of pairwise preferential independence for all pairs of criteria from the set K. This rule may be easily used to explain the result of comparison of real alternatives from the set A. In Figure 4 an example of such an explanation is given. In Appendix D an example of implementation of ZAPROS-LM for a simple case of three alternatives with three criteria is given.

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   ALTERNATIVE  a010 (vector   12121)
               IS MORE PREFERABLE THAN
   ALTERNATIVE  a008 (vector   22211)
         because as a result of the interview it is stated that:

value 1 upon criterion 1 (alt. a010) IS MORE PREFERABLE THAN
value 2 upon criterion 1 (alt. a008);

value 2 upon criterion 2 (alt. a010) IS EQUAL TO
value 2 upon criterion 2 (alt. a008);

value 1 upon criterion 3 (alt. a010) IS EQUAL TO
value 1 upon criterion 4 (alt. a008);

value 2 upon criterion 4 (alt. a010) IS MORE PREFERABLE THAN
value 2 upon criterion 3 (alt. a008);

value 1 upon criterion 5 (alt. a010) IS EQUAL TO
value 1 upon criterion 5 (alt. a008);

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Figure 4. Possible explanations of comparisons on the basis of JOS

7. Verification of the preferential independence of criteria

So, to effectively use the information obtained from a DM it is necessary to have preferential independence of all pairs of criteria (Keeney, 1974). In practical problems we must check if this axiom is not violated in DM's preferences.

The problem of checking this axiom (as well as checking many other axioms of multiattribute utility theory) has no simple solution. In reality, the necessity to use this axiom results from the desire to construct an effective decision rule on the basis of relatively small amount of rather simple information about DM's preferences (the effectiveness of the decision rule means its possibility to guarantee rather high level of compatibility for real alternatives). On the other hand, the full-scale check of DM's preferences implies the need for a DM to carry out a large number of pairwise comparisons. So, the point is to make not a full-scale but sufficient check of DM's preferences to satisfy the axiom's conditions. The following approach is proposed.

Let us recall the list L_2 of vectors from the set Y near the second reference situation. Analogously to the relation R_1 we are able to build the relation of linear quasi-order R_2 on the set L_2. This relation may be used to make some verification of the preferential independence of criteria.

The list L_2 contains vectors with all values but one, equal to the worst ones and with one value at the best level. So, there is a possibility to compare relations between pairs of vectors near two reference situations of the following type:

L_1:

$$y_i = (x_{11}, x_{21}, \dots, x_{(s-1)1}, x_{sn_s}, x_{(s+1)1}, \dots, x_{(t-1)1}, x_{t1}, x_{(t+1)1}, \dots, x_{Q1}),$$

$$y_j = (x_{11}, x_{21}, \dots, x_{(s-1)1}, x_{s1}, x_{(s+1)1}, \dots, x_{(t-1)1}, x_{tn_t}, x_{(t+1)1}, \dots, x_{Q1}).$$

L_2:

$$y_i' = (x_{1n_1}, x_{2n_2}, \dots, x_{(s-1)n_{s-1}}, x_{sn_s}, x_{(s+1)n_{s+1}}, \dots, x_{(t-1)n_{t-1}}, x_{t1}, x_{(t+1)n_{t+1}}, \dots, x_{Qn_Q}),$$

$$y_j' = (x_{1n_1}, x_{2n_2}, \dots, x_{(s-1)n_{s-1}}, x_{s1}, x_{(s+1)n_{s+1}}, \dots, x_{(t-1)n_{t-1}}, x_{tn_t}, x_{(t+1)n_{t+1}}, \dots, x_{Qn_Q}).$$

Both pairs of vectors differ only in components upon criteria s and t. So, pairs differ from one another only in values of equal components. Therefore, if criteria s and t are preferentially independent, the preference in the pairs (y_i, y_j) and (y_i', y_j') has to be the same. So, there is a possibility to carry out some justification of the axiom on the basis of the information obtained near two references situations.

Let us emphasize that though such justification is a very limited one, the violation of this condition rather clearly proves the violation of independence and the necessity of additional analysis of the situation (see later), as all these relations have been thoroughly checked during comparisons near two reference situations. Additionally, let us note that the selected reference situations differ to a very large extent, so the correspondence of the results obtained near them, may be considered to be stable and for all intermediate situations.

It may be shown also that we can use relations R_1 and R_2 for more comprehensive verification of criteria independence. Statements 3, 4, and 5 with the Corollary form the basis for it.

It is easy to prove that the introduced rule for comparison of vectors from Y may be modified for implementation on the basis of the relation R_2.

Statement 3. *If each pair of criteria from K ($Q \geq 3$) does not depend preferentially on criteria, then vector $y_i = (y_{i1}, y_{i2}, \dots, y_{iQ}) \in Y$ is not less preferable for a DM than vector $y_j = (y_{j1}, y_{j2}, \dots, y_{jQ}) \in Y$ if $\forall$ criterion $s \in K$, $\exists$ criterion $t(s) \in K$ such that*

$$(x_{1n_1}, x_{2n_2}, \dots, x_{(s-1)n_{s-1}}, y_{is}, x_{(s+1)n_{s+1}}, \dots, x_{Qn_Q})$$
$$R_2\ (x_{1n_1}, x_{2n_2}, \dots, x_{(t-1)n_{t-1}}, y_{jt}, x_{(t+1)n_{t+1}}, \dots, x_{Qn_Q}),$$

and if criteria s, $q \in K$ are such that $s \neq q$, then $t(s) \neq t(q)$.

Proof. This statement may be proved similarly to the previous one.

There is also a possibility to use R_1 and R_2 simultaneously for comparison of vectors from Y.

Statement 4. *If each pair of criteria from K* ($Q \geq 3$) *does not depend preferentially on other criteria, then vector* $y_i = (y_{i1}, y_{i2}, \ldots, y_{iQ}) \in Y$ *is not less preferable for a DM than vector* $y_j = (y_{j1}, y_{j2}, \ldots, y_{jQ}) \in Y$ *if*:
1) $\forall$ *criterion* $s \in K_1 \subset K$, $\exists$ *criterion* $t(s) \in K_1$ *such that*

$$(x_{11}, x_{21}, \ldots, x_{(s-1)1}, y_{is}, x_{(s+1)1}, \ldots, x_{Q1})\ R_1\ (x_{11}, x_{21}, \ldots, x_{(t-1)1}, y_{jt}, x_{(t+1)1}, \ldots, x_{Q1}),$$

2) $\forall$ *criterion* $s \in K_2 = K \backslash K_1$, $\exists$ *criterion* $t(s) \in K_2$ *such that*

$$(x_{1n_1}, x_{2n_2}, \ldots, x_{(s-1)n_{s-1}}, y_{is}, x_{(s+1)n_{s+1}}, \ldots, x_{Qn_Q})$$
$$R_2\ (x_{1n_1}, x_{2n_2}, \ldots, x_{(t-1)n_{t-1}}, y_{jt}, x_{(t+1)n_{t+1}}, \ldots, x_{Qn_Q}),$$

and if criteria s, $q \in K$ *are such that* $s \neq q$, *then* $t(s) \neq t(q)$.

Proof. Given in Appendix E.

It is clear that possible differences in comparison of the same pairs of vectors from Y on the basis of the JOS built near two reference situations, are caused only by the information about DM's preferences presented in relations R_1 and R_2. Let us prove the following statement.

Statement 5. *Let the comparison of vectors* $(y_i, y_j) \in Y$ *on the basis of the relation* R_1 *and the relation* R_2 *be different. Then there always exist criteria s and t for which*

$$(x_{11}, x_{21}, \ldots, x_{(s-1)1}, y_{is}, x_{(s+1)1}, \ldots, x_{Q1})\ R_1\ (x_{11}, x_{21}, \ldots, x_{(t-1)1}, y_{jt}, x_{(t+1)1}, \ldots, x_{Q1}),$$

but

$$(x_{1n_1}, x_{2n_2}, \ldots, x_{(t-1)n_{t-1}}, y_{jt}, x_{(t+1)n_{t+1}}, \ldots, x_{Qn_Q})$$
$$R_2\ (x_{1n_1}, x_{2n_2}, \ldots, x_{(s-1)n_{s-1}}, y_{is}, x_{(s+1)n_{s+1}}, \ldots, x_{Qn_Q}).$$

Proof. Given in Appendix F.

This statement allows us to carry out the check of the preferential independence of criteria on the basis of R_1 and R_2 due to the corollary below.

Definition 13. Let us call a pair of vectors $(y_i', y_j') \in L_2$ analogous to the pair of vectors $(y_i, y_j) \in L_1$ if:

$$y_i = (x_{11}, x_{21}, \ldots, x_{(s-1)1}, y_{is}, x_{(s+1)1}, \ldots, x_{Q1}),$$
$$y_j = (x_{11}, x_{21}, \ldots, x_{(t-1)1}, y_{jt}, x_{(t+1)1}, \ldots, x_{Q1}),$$
$$y_j' = (x_{1n_1}, x_{2n_2}, \ldots, x_{(t-1)n_{t-1}}, y_{jt}, x_{(t+1)n_{t+1}}, \ldots, x_{Qn_Q}),$$
$$y_i' = (x_{1n_1}, x_{2n_2}, \ldots, x_{(s-1)n_{s-1}}, y_{is}, x_{(s+1)n_{s+1}}, \ldots, x_{Qn_Q}).$$

Corollary. *If comparisons between all analogous pairs of vectors near two reference situations are the same, meaning that if*

$$(x_{11}, x_{21}, \ldots, x_{(s-1)1}, y_{is}, x_{(s+1)1}, \ldots, x_{Q1})\ R_1\ (x_{11}, x_{21}, \ldots, x_{(t-1)1}, y_{jt}, x_{(t+1)1}, \ldots, x_{Q1}),$$

then

$$(x_{1n_1}, x_{2n_2}, \ldots, x_{(s-1)n_{s-1}}, y_{is}, x_{(s+1)n_{s+1}}, \ldots, x_{Qn_Q})$$
$$R_2\ (x_{1n_1}, x_{2n_2}, \ldots, x_{(t-1)n_{t-1}}, y_{jt}, x_{(t+1)n_{t+1}}, \ldots, x_{Qn_Q}),$$

then it is impossible to detect violations of preferential independence of criteria on the basis of the obtained information.

Proof. Evident, if the above-marked possibilities to detect violations of the axiom about preferential independence of criteria are considered.

This gives us the opportunity to find out pairs of dependent criteria by analyzing comparisons near two reference situations (if there are any). But even if there are some violations of the introduced conditions, we can evaluate the consequences for the comparison of real alternatives from the set A.

If according to one rule alternatives can be compared and according to the other we are not able to compare them, then this is not a contradictory situation. We have just enlarged compatibility of alternatives on the basis of additional information from a DM about comparison of vectors from L_2.

Only if the results of comparison contradict each other, then this is connected with violation of criteria independence and the alternatives are to be considered incomparable.

So, we are able to estimate the number of alternatives' pairs which it will be possible to compare additionally if we analyze the dependency thoroughly. Analysis and elicitation of dependent criteria and also procedures for reformulation of the initial task in this case (Larichev and Moshkovich, 1991) are rather labour-consuming. So, a DM is able to evaluate if he (or she) wants to spend rather large amount of time and effort, knowing the maximum of additional information about comparison of alternatives which it is possible to obtain as a result.

8. Conclusion

The proposed method aims the construction of a rank-ordering of the set of alternatives on the basis of a decision-maker's preferences. It is focused on elicitation of the DM's preferences in a qualitative (ordinal) form and on the implementation of a logical transition to a decision rule for comparison of real alternatives. The criteria with verbal scales may be used, where necessary. There is a possibility to organize a reasonable interview with a DM to obtain information about his (her) preference system, to detect possible errors in his (her) responses and to correct them on the basis of their analysis in a dialogue with a DM.

So, the advantages of the method ZAPROS-LM are:

- use of simple and understandable information (judgements) from a DM;
- provision of a thorough consistency check for the involved assumptions (transitivity and independence);
- easy explanations of the results;
- theoretical validation.

But this method does not guarantee the comparability of all pairs of real alternatives. In this case we have only partial order and additional analysis is needed to make the ranking (see Larichev and Moshkovich, 1991, for more details). But let us recall that this technique is proposed for the tasks with large number of alternatives. In these circumstances, a number of questions is considered to be reasonable, and 'good' partial order may be quite satisfactory for the decision maker, especially as the result is transparent and may be easily interpreted.

The method is computerized and is used in practical tasks of decision making.

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References

Aho, A.V., Hopcroft, J.E., and Ullman, J.D. (1974), *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, MA.

Clarckson, G.P. (1962), *Portfolio Selection: A Simulation of Trust Investment*, Prentice-Hall, Englewood Cliffs, NJ.

Fishburn, P.C. (1970), *Utility Theory for Decision Making*, Wiley, New York.

Furems, E.M., and Moshkovich, H.M. (1984), "Ordering of vectors for the task of portfolio selection", in: O.I. Larichev (ed.), *Procedures for Estimating Multiattribute Alternatives*, VNIISI Press 9, Moscow, 41–53 (in Russian).

Furems, E.M., Vasunin, G.N., Larichev, O.I., and Chernov, Yu.G. (1982), "Problems of bin packing with multiple criteria", in: O.I. Larichev (ed.), *Problems and Methods of Decision Making in Organizations*, VNIISI Press, Moscow, 84–91 (in Russian).

Garey, M.R., and Johnson, D.S. (1979), *Computers and Intractability*, Freeman, San Francisco, CA.

Gnedenko, L.S., Larichev, O.I., Moshkovich, H.M., and Furems, E.M. (1986), "Procedure for construction of a quasi-order on the set of multiattribute alternatives on the base of reliable information about the decision maker's preferences", *Avtomatica i telemekhanika* 9, 104–113 (in Russian).

Hoffman, P.J., Slovic, P., and Rorer, L.G. (1968), "Analysis-of-variance model for assessment of of configural cue utilization in clinical judgment", *Psychological Bulletin* 69, 338–349.

Keeney, R.L. (1974), "Multiplicative utility functions", *Operations Research* 22, 22–34.

Keeney, R., and Raiffa, H. (1976), *Decisions with Multiple Objectives: Preference and Value Tradeoffs*, Wiley, New York.

Kendall, M. (1969), *Rank Correlations*, Academic Press, New York.

Kirkwood, C.W., and Sarin, R.K. (1985), "Ranking with partial information: A method and an application", *Operations Research* 33/1, 38–48.

Larichev, O.I. (1979), *Science and Art of Decision making*, Nauka, Moscow.

Larichev, O.I. (1982), A method for evaluating R&D proposals in large research organizations", Collaborative Paper CP-82-75, IIASA, Laxenburg, Austria.

Larichev, O.I., and Moshkovich, H.M. (1988), "Limits to decision making ability in direct multiattribute alternative evaluation", *OBHDP* 42, 217–233.

Larichev, O.I., and Moshkovich, H.M. (1991), "ZAPROS: A method and a system for ordering of multiattribute alternatives on the basis of decision-maker's preferences", Preprint, VNIISI Press, Moscow (in English).

Larichev, O.I., Polyakov, O.A., and Nikiforov, A.D. (1987), "Multicriterion linear programming problems (analytical survey)", *Journal of Economical Psychology* 8, 389–407.

Larichev, O.I., Zuev, Yu.A., and Gnedenko, L.S. (1974), "Method for classification of applied R&D projects", in: S.V. Emelyanov (ed.), *Perspective Applied R&D Planning*, Nauka, Moscow, 28–57 (in Russian).

Larichev, O.I., Moshkovich, H.M., and Rebrik, S. (1988), "Systematic research into human behavior in multiattribute object classification problems", *Acta Psychologica* 68, 171–182.

Larichev, O.I., Zuev, Yu.A., and Gnedenko, L.S. (1979), "Method ZAPROS (Closed Procedures near Reference Situations) for solution of ill-structured problems with multiple criteria", Preprint, VNIISI Press, Moscow (in Russian).

Mirkin, B.G. (1974), *Problems of Group Choice*, Nauka, Moscow (in Russian).

Montgomery, H. (1977), "A study of intransitive preferences using a think aloud procedure", in: H. Jungerman and G. de Zeeuw (eds.), *Decision Making and Change in Human Affairs*, Reidel, Dordrecht.

Moshkovich, H.M. (1988), "Interactive system ZAPROS (for ordering of multiattribute alternatives on the basis of decision maker's preferences)", in: VNIISI Press 11, Moscow, 13–21 (in Russian).

Ore, O. (1962), "Theory of graphs", American Mathematical Society, Rhode Island.

Ozernoy, V.M., and Gaft, M.G. (1978), "Multicriteria problems of decision making", in: *Decision Making Problems*, Mashinostroenie Press, Moscow, 14–47 (in Russian).

Payne, J.W. (1976), "Task complexity and contingent processing in decision making: An information search and protocol analysis", *OBHP* 16, 366–387.

Rivett, P. (1977), "The dog that did not bark", *Engineering Economics* 22/4, 298–300.

Roy, B. (1985), *Méthodologie Multicritère d'Aide à la Décision*, Economica, Paris.

Russo, J.E., and Rosen, L.D. (1975), "An eye fixation analysis of multiattribute choice", *Memory and Cognition* 3, 267–276.

Svenson, O. (1979), "Process descriptions of decision making", *Organization Behavior and Human Performance* 23, 86–112.

Svenson, O. (1983), "Decision rules and information processing in decision making", in: L. Sjoberg, T. Tyszka and J.A. Wise (eds.), *Human Decision Making*, Doxa, Bodafors.

Tversky, A. (1969), "Intransitivity of preferences", *Psychological Review* 76, 31–48.

Watson, S.R., and Buede, D.M. (1987), *Decision Synthesis: The Principles and Practice of Decision Analysis*, Cambridge University Press, Cambridge.
Wilson, R. (1972), *Introduction to Graph Theory*, Oliver & Boyd, Edinburgh.
Zuev, Yu.A., Larichev, O.I., Filippov, V.A., and Chuev, Yu.V. (1980), "Problems of evaluation of R&D projects", *Vestnik Akademii Nauk SSSR* 8, (in Russian).

Appendix A. Criteria for job evaluation

Criterion 1. Salary:
1. The salary is rather high.
2. The salary is on the average level.
3. The salary is a bit lower the average level.

Criterion 2. Job location:
1. Location of the job is almost ideal.
2. Location of the job is some distance from the ideal.
3. Job is located far away from where you want.

Criterion 3. Type of the job position:
1. Type of the job position is almost ideal.
2. Type of the job position is good enough (in field).
3. Type of the job position is not appropriate.

Criterion 4. Possibilities for training:
1. There are nice possibilities for training.
2. There are normal possibilities for training.
3. There are almost none possibilities for training.

Criterion 5. Possibilities for promotion:
1. There are good possibilities for promotion.
2. There are moderate possibilities for promotion.
3. There are almost no possibilities for promotion.

Appendix B. Evaluation of the number of questions to the DM

While comparing vectors near reference situations, we need to fill in a matrix A_1 of pairwise comparisons of size $N_1 \times N_1$, where $N_1 = \Sigma_{q=1}^{Q}(n_q - 1)$. Let us estimate the number of required responses from a DM in the worst case for Q criteria with $n_q = m$, $\forall q \in K$.

In general we are to fill in $Q \times \frac{1}{2}(Q-1)$ submatrices for each two criteria of size $(m-1) \times (m-1)$ (that is the over-diagonal part of the initial matrix M). So, initially we have to fill in $Q \times (Q-1) \times (m-1)^2$ elements.

If we ask the DM to compare vectors y_1 and y_2 from L_1 with $y_{1s} = x_{si} \neq x_{s1}$ and $y_{2t} = x_{ti} \neq x_{t1}$ $(i \leq m)$, then if according to the DM $(y_1, y_2) \in P_{\mathrm{DM}}$ or $(y_2, y_1) \in P_{\mathrm{DM}}$, then using the relation P^0, we are able to say that this relation is maintained for vectors from L_1 with x_{tj}, $j > i$, in the first case and x_{sj}, $j > i$, in the second. So, if $i = 2$, we are able to fill in $m-1$ elements in the matrix M in the case of strict preference in the pair of vectors from L_1. If the DM responds that $(y_1, y_2) \in I_{\mathrm{DM}}$, we are able to define the relation for vectors with x_{sj} and x_{tj}, $j > i$. Therefore, in any case by one respond we are able to fill in $m - i + 1$ elements $(i = 1, 2, \ldots, m)$. Asking the DM to compare vectors corresponding to the diagonal elements of a sub matrix (for two criteria), the number of questions to the DM will be equal to

$(m-1)^2-(m-1)\times\frac{1}{2}(m-2)$. For the whole matrix M the number of questions C to the DM in the worst case will be equal to

$$C=\frac{1}{2Q}(Q-1)\left[(m-1)^2-\tfrac{1}{2}(m-1)(m-2)\right]=\frac{1}{4Q}(Q-1)m(m-1).$$

This leads us to the following numbers: $Q=5$, $m=3$, $C_1=30$; $Q=6$, $m=3$, $C_1=36$; $Q=5$, $m=4$, $C_1=60$.

We must realize that in real tasks this number may be less, as relations of indifference give more information and the transitive closure may fill in elements of other sub matrices. But if we consider additional questions in the case of contradictory responses and the construction of the JOS near the second reference situation, this number may become rather large, though explainable and the questions do not require much time for consideration.

Appendix C

Proof of Statement 2. [1] According to Theorem 3.7 (Keeney and Raiffa, 1976), if each pair of criteria does not depend preferentially on other criteria, then all criteria are mutually preferentially independent.

According to Theorem 3.6 (Keeney and Raiffa, 1976), in this case there exists an additive value function

$$v(y_i)=\sum_{q=1}^{Q}v_q(y_{iq})\quad\text{for criteria of the set } K.$$

Let there be $y_i,\ y_j\in Y$: $y_i=(y_{i1}, y_{i2},\dots, y_{iQ})$ and $y_j=(y_{j1}, y_{j2},\dots, y_{jQ})$. Then according to the initial condition, $\forall y_{is}$, $\exists$ such y_{jt}, which with all the other best values upon the rest criteria is not less preferable that this one. Let it be the following:

1) $(y_{i1}, x_{21},\dots, x_{Q1})\ R_1\ (x_{11}, y_{j2}, x_{31},\dots, x_{Q1})$;

2) $(x_{11}, y_{i2}, x_{31},\dots, x_{Q1})\ R_1\ (x_{11}, x_{21}, y_{j3}, x_{41},\dots, x_{Q1})$;

$\vdots$

Q) $(x_{11}, x_{21},\dots, x_{(Q-1)1}, y_{iQ})\ R_1\ (y_{j1}, x_{21},\dots, x_{Q1})$.

Expression 1) means that

$$v_1(y_{i1})+v_2(x_{11})+\cdots+v_Q(x_{Q1})\geq v_1(x_{11})+v_2(y_{j2})++v_3(x_{31})+\cdots+v_Q(x_{Q1})$$

and as a result, $v_1(y_{i1})+v_2(x_{21})\geq v_1(x_{11})++v_2(y_{j2})$.

Expression 2) means that

$$\begin{aligned}&v_1(x_{11})+v_2(y_{i2})+v_3(x_{31})+\cdots+v_Q(x_{Q1})\\&\quad\geq v_1(x_{11})+v_2(x_{21})+v_3(y_{j3})+v_4(x_{41})+\cdots+v_Q(x_{Q1})\end{aligned}$$

and as a result $v_2(y_{i2})+v_3(x_{31})\geq v_2(x_{21})++v_3(y_{j3})$ and so on. Let us add the right and left parts of the resulting inequalities, obtaining

$$\begin{aligned}&v_1(y_{i1})+v_2(x_{21})+v_2(y_{i2})+v_3(x_{31})+\cdots+v_{Q-1}(y_{iQ-1})+v_Q(x_{Q1})+v_Q(y_{iQ})+v_1(x_{11})\\&\quad\geq v_1(x_{11})+v_2(x_{21})+v_2(y_{j2})+v_3(y_{j3})+\cdots+v_{Q-1}(x_{(Q-1)1})+v_Q(y_Q)+v_Q(x_{Q1})+v_1(y).\end{aligned}$$

[1] As usual we assume that the value function exists and exhibits the properties of continually and differentially (see e.g., Fishburn, 1970), though we do not implicitly use these properties in the process of preference elicitation.

Table D.1

Jobs	Criteria		
	Salary	Job location	type of job position
Proposal1	Rather high	Some distance	Inappropriate
Proposal2	Average	Far away	Ideal
Proposal3	Lower	Ideal	good enough

Eliminating common elements we have

$$\nu_1(y_{i1}) + \nu_2(y_{i2}) + \cdots + \nu_Q(y_{iQ}) \geq \nu_1(y_{j1}) + \nu_2(y_{j2}) + \cdots + \nu_Q(y_{jQ}).$$

So, $\nu(y_i) \geq \nu(y_j)$, which is what was required to be proven.

Appendix D. Analysis of an example

Let us illustrate the proposed method ZAPROS-LM on a simple example. A young man, about to graduate from the college, receives concrete job offers from three companies, whose sites he has visited. He plans to accept one of the three jobs, but is not sure which he prefers most. He uses three main criteria to characterize each proposal: Salary, Job location, and Type of job position (the first three criteria in the list in Appendix A). He estimated the proposals upon these criteria as shown in the Table D.1.

So, we can say that we need to compare vectors $b_1 = (1, 2, 3)$, $b_2 = (2, 3, 1)$ and $b_3 = (3, 1, 2)$ which correspond to Proposal1, Proposal2 and Proposal3. Let us try to do this through construction of the joint ordinal scale for criteria the considered three criteria.

First we form list of vectors near the first reference situation $L_1 = \{211, 311, 121, 131, 112, 113\}$. So, we have six vectors in this list and need to fill in the matrix of pairwise comparisons for them (further we will show only the elements above the diagonal and use: 1 – to mark that the element in the row is more preferable than element in the column; 2 – to show equal preferable elements; 3 – to mark that the element in the row is less preferable than the element in the column; 0 means that these elements are not compared).

In Figure D.1a you can see the initial matrix with comparisons made upon the relation of dominance P^0. In figure D.1b you can see the same matrix, but after the first comparison fulfilled by the DM: he

	1 1 2	1 1 3	1 2 1	1 3 1	2 1 1	3 1 1
211	2	1	0	0	0	0
311		2	0	0	0	0
121			2	1	0	0
131				2	0	0
112					2	1
113						2

a)

	1 1 2	1 1 3	1 2 1	1 3 1	2 1 1	3 1 1
211	2	1	1	1	0	0
311		2	0	0	0	0
121			2	1	0	0
131				2	0	0
112					2	1
113						2

b)

	1 1 2	1 1 3	1 2 1	1 3 1	2 1 1	3 1 1
211	2	1	1	1	1	1
311		2	3	3	3	1
121			2	1	3	1
131				2	3	1
112					2	1
113						2

c)

	3 3 2	3 3 1	3 2 3	3 1 3	2 3 3	1 3 3
233	2	3	1	?	1	?
133		2	?	3	?	1
323			2	3	3	?
313				2	?	1
332					2	?
331						2

d)

Figure D.1. Matrix of pairwise comparisons near the first reference situation

Table D.2

Jobs	Vectors		
	Initial	Upon JOS	Ordered upon JOS
Proposal1	(1, 2, 3)	(1, 4, 7)	(1, 4, 7)
Proposal2	(2, 3, 1)	(2, 5, 1)	(1, 2, 5)
Proposal3	(3, 1, 2)	(6, 1, 3)	(1, 3, 6)

prefers vector 211 to vector 121. The underlined 1 marks the conclusion made upon transitivity: $(211, 121) \in P_{DM}$ and $(121, 131) \in P^0$. This implies that $(211, 131) \in P_{DM}$. Now the DM is to compare vectors 211 and 112. In figure D.1c you can see the entirely fulfilled matrix (underlined numbers mean that they are defined upon transitivity).

As we can see, we needed five comparisons from the DM to fill in the matrix, five comparisons were carried out on the basis of transitivity, but only two of them were confirmed more than once. So, we need to carry out three more comparisons to check the information (121 with 113; 131 with 112; 131 with 113). If the result is the same we can construct the joint ordinal scale on the basis of the matrix at Figure D.1c (we will not analyze here the independence of criteria, but just assume it – the elements of the matrix near the second reference situation which are to be the same in such case are shown in Figure D1d).

According to the matrix in Figure D.1c we can rank-order vectors of the list L_1 as follows: 211, 112, 121, 131, 311, 113. Using this scale, let us try to compare the initial three alternatives: the first value upon any criterion has the first rank, so, the second value upon the first criterion (211) has the second rank, the second value upon the third criterion (112) has the third rank and so on.

Let us change in each vector, describing real proposals the numbers of values upon criteria for ranks in the joint ordinal scale. The result is represented in the third column of Table D.2. After that we rewrite these vectors with values in the descending order (see column 4 of Table D.2).

Now we see that the newly obtained vectors may be ordered according to the dominance relation: the vector for Proposal2 is more preferable than the vector for Proposal1 and Proposal3; the vector for Proposal3 is more preferable than the vector for Proposal1. So, we rank-ordered proposals and the second is the most preferable one. This is evident as the dominance shows us a pair of values which can allow us to compare proposals. The corresponding explanations are presented in Figure D.2.

```
==================================================================
                                 I
PROPOSAL2 (vector 231) is more preferable than PROPOSAL1 (vector 123)
   as according to the joint ordinal scale:

value 2 criterion 1 IS MORE PREFERABLE than value 2 criterion 2
value 3 criterion 2 IS MORE PREFERABLE than value 3 criterion 3
value 1 criterion 3 IS EQUAL TO value 1 criterion 1

                                 II
PROPOSAL2 (vector 231) is more preferable than PROPOSAL3 (vector 312)
   as according to the joint ordinal scale:

value 2 criterion 1 IS MORE PREFERABLE than value 2 criterion 3
value 3 criterion 2 IS MORE PREFERABLE than value 3 criterion 1
value 1 criterion 3 IS EQUAL TO value 1 criterion 2

                                 III
PROPOSAL3 (vector 312) is more preferable than PROPOSAL2 (vector 123)
   as according to the joint ordinal scale:

value 3 criterion 1 IS MORE PREFERABLE than value 3 criterion 3
value 1 criterion 2 IS EQUAL TO value 1 criterion 1
value 2 criterion 3 IS MORE PREFERABLE than value 2 criterion 2
==================================================================
```

Figure D.2. Explanations of pairwise comparisons of job proposals

Appendix E

Proof of Statement 4. As in the previous case, the preferential independence of all pairs of criteria from K implies the existence of the additive value function $v(y_i) = \sum_{q=1}^{Q} v_q(y_{iq})$.

Let us have two vectors from Y: $y_i = (y_{i1}, y_{i2}, \ldots, y_{iQ})$ and $y_j = (y_{j1}, y_{j2}, \ldots, y_{jQ})$. Then according to the initial condition, $\forall y_{is}$, $\exists$ such y_{jt}, that condition 1) or condition 2) is fulfilled.

Let us renumerate criteria in such a way that for the first m criteria condition 1) is fulfilled and for all the others condition 2). This means:

$$
\begin{array}{ll}
1) & (y_{i1}, x_{21}, \ldots, x_{Q1}) \; R_1 \; (x_{11}, x_{21}, \ldots, y_{jt(1)}, x_{(t(1)+1)1}, \ldots, x_{Q1}), \\
2) & (x_{11}, y_{i2}, x_{31}, \ldots, x_{Q1}) \; R_1 \; (x_{11}, x_{21}, \ldots, y_{jt(2)}, x_{(t(2)+1)1}, \ldots, x_{Q1}), \\
\vdots & \\
m) & (x_{11}, x_{21}, \ldots, x_{(m-1)1}, y_{im}, x_{(m+1)1}, \ldots, x_{Q1}) \\
 & R_1 \; (x_{11}, x_{21}, \ldots, x_{(t(m)-1)1}, y_{jt(m)}, x_{(t(m)+1)1}, \ldots, x_{Q1}), \\
m+1) & (x_{n_1}, x_{n_2}, \ldots, y_{i(m+1)}, x_{n_{(m+2)}}, \ldots, x_{n_Q}) \; R_2 \; (x_{n_1}, x_{n_2}, \ldots, y_{jt(m+1)}, x_{n_{[t(m+1)+1]}}, \ldots, x_{n_Q}), \\
\vdots & \\
Q) & (x_{n_1}, x_{n_2}, \ldots, x_{n_{Q-1}}, y_{iQ}) \; R_2 \; (x_{n_1}, x_{n_2}, \ldots, y_{jt(Q)}, x_{n_{[t(Q)+1]}}, \ldots, x_{n_Q}).
\end{array}
$$

As in the previous cases, these relations may be presented as inequalities in summed value functions of vectors from the right and left parts. Summing up right and left parts of these inequalities, we shall have:

$$
\sum_{s=1}^{m} \sum_{\substack{q \neq s \\ q \in K}} v_q(x_{q1}) + \sum_{q=1}^{m} v_q(y_{iq}) + \sum_{s=m+1}^{Q} \sum_{\substack{q \neq s \\ q \in K}} v_q(x_{n_q}) + \sum_{q=m+1}^{Q} v_q(y_{iq})
$$
$$
\leq \sum_{s=1}^{m} \sum_{\substack{q \neq t(s) \\ q \in K}} v_q(x_{q1}) + \sum_{s=1}^{Q} v_{t(s)}(y_{jt(s)}) + \sum_{s=m+1} \sum_{\substack{q \neq t(s) \\ q \in K}} v_q(x_{n_q}) + \sum_{s=m+1}^{Q} v_{t(s)}(y_{jt(s)}).
$$

This results in

$$
\sum_{s=1}^{m} \left(\sum_{\substack{q \neq s \\ q \in K}} v_q(x_{q1}) - \sum_{\substack{q \neq t(s) \\ q \in K}} v_q 1) \right) + \sum_{q=1}^{Q} v_q(y_{iq}) + \sum_{s=m+1}^{Q} \left(\sum_{\substack{q \neq s \\ q \in K}} v_q(x_{n_q}) - \sum_{\substack{q \neq t(s) \\ q \in K}} v_q(x_{n_q}) \right)
$$
$$
\leq \sum_{q=1}^{Q} v_q(y_{iq}).
$$

This corresponds to

$$
\sum_{s=1}^{m} \left(v_{t(s)}(x_{t(s)1}) - v_s(x_{s1}) \right) + \sum_{s=m+1}^{Q} \left(v_{t(s)}(x_{n_{t(s)}}) - v_s(x_{n_s}) \right) + \sum_{q=1}^{Q} v_q(y_{iq}) \leq \sum_{q=1}^{Q} v_q(y_{iq}).
$$

Note that the first and the second sums are equal to zero, as according to the initial condition $t(s) \in \{1, 2, \ldots, m\}$ when $s = 1, 2, \ldots, m$, and when $s = m+1, m+2, \ldots, Q$, $t(s) \in \{m+1, m+2, \ldots, Q\}$ and these numbers of criteria are not repeated.

So, $v(y_i) \leq v(y_j)$, what was required to be proved.

Appendix F

Proof of Statement 5. As it is true that $(y_i,\ y_j) \in R_{\mathrm{DM}}$ on the base of relation R_1, it follows that according to the definition, $\forall s \in K$, $\exists t(s) \in K$ such that

$$(1, 1, \ldots, y_{is}, 1, \ldots, 1)\ R_1\ (1, 1, \ldots, y_{jt(s)}, 1, \ldots, 1).$$

As the relation R_2 is a connected one, it follows that

$$(1)\ (x_{n_1}, x_{n_2}, \ldots, x_{n_{s-1}}, y_{is}, x_{n_{s+1}}, \ldots, x_{n_Q})\ R_2\ (x_{n_1}, x_{n_2}, \ldots, x_{n_{t(s)-1}}, y_{jt(s)}, x_{n_{t(s)+1}}, \ldots, x_{n_Q}),$$

or

$$(2)\ (x_{n_1}, x_{n_2}, \ldots, x_{n_{t(s)-1}}, y_{jt(s)}, x_{n_{t(s)+1}}, \ldots, x_{n_Q})\ R_2\ (x_{n_1}, x_{n_2}, \ldots, x_{n_{s-1}}, y_{is}, x_{n_{s+1}}, \ldots, x_{n_q}).$$

If (2) is true, then the statement is proved. So, let for each $s \in \mathrm{K}$ condition (1) be fulfilled. Then according to the definition, $(y_i,\ y_j) \in R_{\mathrm{DM}}$ on the base of the relation R_2 and this contradicts the initial conditions of the statement.